§ 9. Differential equations

§ 9.1 Examples

Example 9.1 (Population growth):

A population grows at a rate proportional to the size of the population (assumption):

t = time

P = the number of individuals in the population

$\Rightarrow \frac{dP}{dt}$ = rate of growth (1)

Assumption $\Rightarrow \frac{dP}{dt} = kP$

where k is the proportionality constant.

We have: $P(t) > 0 \ \forall t$

Thus for $k>0$: $P'(t) > 0 \ \forall t$

Solution (see Example 5.13): $P(t) = Ce^{kt}$

then $P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$

→ family of solutions

for varying C

Putting $t=0$, we get $P(0) = Ce^{k(0)} = C$

$\Rightarrow$ C is the initial population $P(0)$.

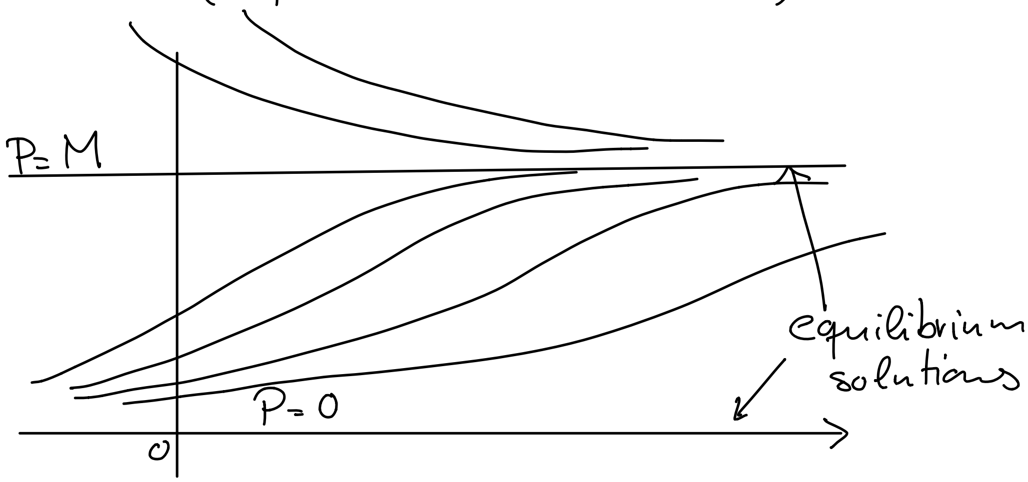
Many populations have a "carrying capacity" M and start decreasing when they hit this number:

- $\frac{dP}{dt} \sim kP$ if P is small $\left(\frac{P}{M} \ll 1\right)$
- $\frac{dP}{dt} < 0$ if $P > M$

$$\Rightarrow \quad \frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \qquad (2)$$

"logistic differential equation"

Constant solutions $P(t) = 0$ and $P(t) = M$ are solutions ("equilibrium solutions").



Definition 9.1 (First order ODE):

A first order "ordinary differential equation" is an equation of the form:

$$F(x, y, y') = 0 \qquad (4)$$

where F is some arbitrary function and y is understood to be an unknown function of x. For example,

$$y' = xy$$

is of the form (4) with $F(x,y,y') = y' - xy$

A function $f(x)$ is called a "solution" of the differential equation if (4) is satisfied with $y = f(x)$ and $y' = f'(x)$.

Example 9.2:

Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

Solution:

We use the quotient rule to differentiate:

$$y' = \frac{(1-ce^t)(ce^t) - (1+ce^t)(-ce^t)}{(1-ce^t)^2}$$

$$= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1-ce^t)^2} = \frac{2ce^t}{(1-ce^t)^2}$$

The right side of the differential equation becomes

$$\frac{1}{2}(y^2-1) = \frac{1}{2}\left[\left(\frac{1+ce^t}{1-ce^t}\right)^2 - 1\right] = \frac{1}{2}\left[\frac{(1+ce^t)^2-(1-ce^t)^2}{(1-ce^t)^2}\right]$$

$$= \frac{1}{2}\ \frac{4ce^t}{(1-ce^t)^2} = \frac{2ce^t}{(1-ce^t)^2}$$

Therefore, for every value of c, the given function is a solution of the differential equation.

Example 9.3 (Initial value problem):

Find a solution of the differential equation $y' = \frac{1}{2}(y^2-1)$ that satisfies the initial condition $y(0) = 2$.

Solution:

Substituting the values $t=0$ and $y=2$ into the formula

$$y = \frac{1+ce^t}{1-ce^t}$$

from Example 7.1.3, we get

$$2 = \frac{1+ce^0}{1-ce^0} = \frac{1+c}{1-c}$$

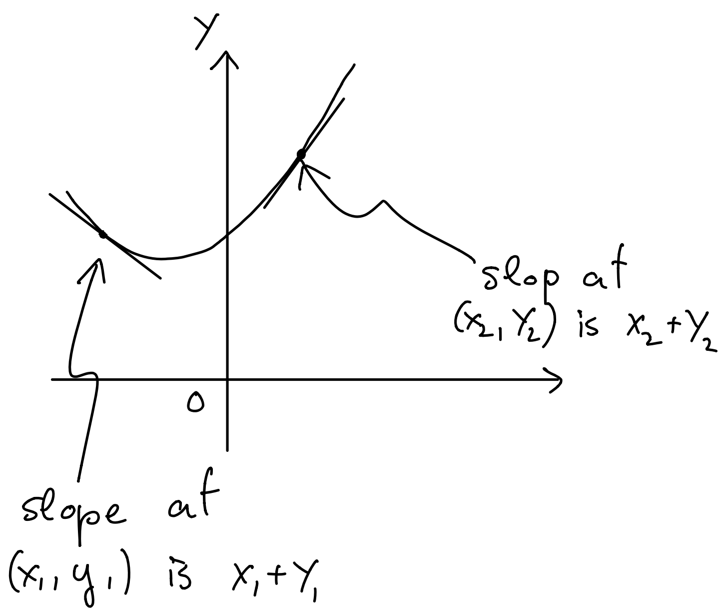
$c = \frac{1}{3}$ solves this equation $\Rightarrow y = \frac{1+\frac{1}{3}e^t}{1-\frac{1}{3}e^t} = \frac{3+e^t}{3-e^t}$

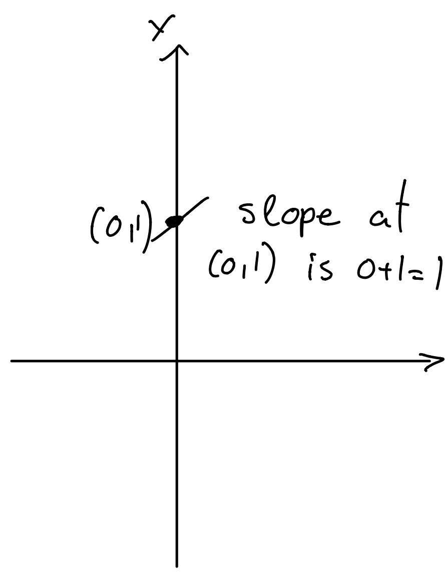
§9.2 Existence and Uniqueness

<u>Example 9.4</u> (Direction Fields):

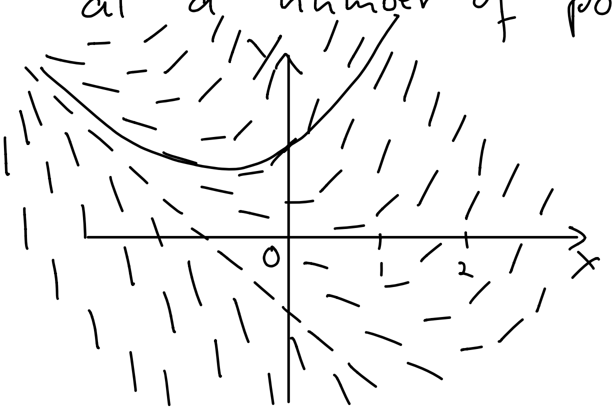
Suppose we are asked to sketch the graph of the solution of the initial value problem

$$y' = x + y \quad , \quad y(0) = 1$$





We can also draw short line segments at a number of points (x, y) with slope $x+y$:



This is called a "direction field". For instance, the line segment through (1,2) has slope 1+2 = 3.

Definition 9.2:

Suppose we have a first-order differential equation of the form

(1) $\quad y' = F(x, y)$ $\quad$ "normal form"

where $F(x, y)$ is a continuous function of both its arguments simultaneously in some domain D of the x-y plane.

The slope of the solution curve through (x, y) is $F(x, y)$. Indicating such slopes on the x-y-plane in terms of line segments gives a direction field.

A solution or "integral" of (1) over th interval $x_0 \le x \le x_1$ is a single-valued function $y(x)$ with a continuous first derivative $y'(x)$ defined on $[x_0, x_1]$ such that for $x_0 \le x \le x_1$:

i) $(x, y(x))$ is in D, whence $F(x, y(x))$ is defined

ii) $y'(x) = F(x, y(x))$

__Definition 9.3__ (Lipschitz condition):

A function $F(x,y)$ defined on a domain D is said to satisfy Lipschitz conditions with respect to y for the constant $K>0$ if for every x, y_1, y_2 such that $(x,y_1), (x,y_2)$ are in D:

$$|F(x,y_1)-F(x,y_2)| \leqslant K|y_1-y_2| \qquad (2)$$

__Theorem 9.1:__

If the function $F(x,y)$ is continuous in both of its arguments and satisfies the Lipschitz condition for y on the rectangle

$$(R): \ |x-x_0| \leq a, \ |y-y_0| \leq b,$$

then there exists a unique solution to the initial value problem

$$y' = F(x,y), \quad y(x_0) = y_0$$

defined on the interval $|x-x_0|<h$ where

$$h = \min(a, b/M), \quad M = \max|F(x,y)|, \ (x,y)\in(R).$$

Remark 9.1:

In other words, two integral curves cannot meet or intersect at any point of R. Observe that without the additional requirement of a Lipschitz condition uniqueness need not follow. For consider the differential equation

$$\frac{dy}{dx} = y^{\frac{1}{3}}$$

$F(x, y) = y^{\frac{1}{3}}$ is continuous at $(0,0)$. But there are two solutions passing through $(0,0)$, namely

i) $y = 0$

ii) $$\begin{cases} y = \left(\frac{2}{3}x\right)^{3/2} & x \geq 0 \\ y = 0 & x \leq 0 \end{cases}$$

$y^{\frac{1}{3}}$ does <u>not</u> satisfy the Lipschitz condition at $y=0$:

for $y_1 = \delta$, $y_2 = -\delta$, $\left|\frac{f(y_1) - f(y_2)}{y_1 - y_2}\right| = \frac{1}{\delta^{2/3}}$

which is unbounded for δ arbitrarily small.

§ 9.3 Solution Methods :

Definition 9.4 (linear ODEs):

A general first order ODE is called "linear" if it can be written in the form

$$a_0(x)y' + a_1(x)y = b(x) \qquad (1)$$

where $a_0(x)$, $a_1(x)$, $b(x)$ are continuous functions of x on some interval I.

Remark 9.2:

Equation (1) can be brought to normal form if $a_0(x) \neq 0$ on I:

$$y' = q(x) - p(x)y \qquad (2)$$

where $p(x) = \frac{a_1(x)}{a_0(x)}$, $q(x) = \frac{b(x)}{a_0(x)}$.

Definition 9.5 (linear homogeneous eq.):

A linear first order ODE is homogeneous if $q(x) = 0$, namely,

$$\frac{dy}{dx} + p(x)y = 0 \qquad (3)$$

Equation (2) is then called "linear inhomogeneous".

Let now $p(x)$ be continuous on an interval $I \subset \mathbb{R}$ and let $J \subset \mathbb{R}$ be closed. Then $F(x,y) := -p(x)y$ is continuous in x and satisfies the Lipschitz condition in y on the rectangular area $R = I \times J$.

$\overset{\text{Th. 9.1}}{\Longrightarrow}$ $\exists$ unique solution to (3) with

$$y(x_0) = y_0 \quad \text{for } (x_0, y_0) \in I \times J.$$

<u>Proposition 9.1:</u>

The solutions to (3) are parametrized by

$$y = \pm C_1 e^{-\int p(x)dx}$$

where $C_1 \in \mathbb{R}$ is fixed through the initial value.

<u>Proof:</u>

Equation (3) can be rewritten as

$$\frac{y'(x)}{y} = \frac{d}{dx} \log|y(x)| = -p(x).$$

Integration then gives

$$\log|y(x)| = -\int p(x)dx + C,$$

or $$y = \pm C_1 e^{-\int p(x)dx}, \quad C_1 = e^C > 0. \qquad \square$$

We next want to solve the general case, namely

$$\frac{dy}{dx} + p(x)\,y = q(x) \quad \text{(inhomogeneous case)}$$

Multiplication with a factor $\mu(x) \neq 0$ gives:

$$\mu(x)\,y'(x) + \mu(x)\,p(x)\,y(x) = \mu(x)\,q(x). \qquad (4)$$

Now choose $\mu(x)$ such that

$$\mu'(x) = p(x)\,\mu(x) \;,$$

i.e. $\mu(x)$ satisfies the linear homogeneous eq. (solution given by Prop. 9.1)

$\Rightarrow$ eq. (4) becomes

$$\frac{d}{dx}\Big[\mu(x)\,y(x)\Big] = \mu(x)\,q(x) \qquad (5)$$

Integration gives

$$\mu(x)\,y = \int \mu(x)\,q(x)\,dx + C$$

with C an arbitrary constant. Solving for y, we then arrive at the following:

<u>Proposition 9.2:</u>

$$y = \frac{1}{\mu(x)} \int \mu(x)\,q(x)\,dx + \frac{C}{\mu(x)} = \underbrace{y_p(x)}_{\text{"particular"}} + \underbrace{y_H(x)}_{\text{"homogeneous"}} \qquad (6)$$

is a solution to the linear inhomogeneous ODE (2) if $\mu(x)$ is a solution to the homogeneous ODE $\mu'(x) = p(x)\,\mu(x)$.

Remark 9.3:

Again, by Th. 9.1, the solution (6) is unique once C is fixed using the initial value $y(x_0) = y_0$.

Example 9.5:

$y' + xy = x$. This is a linear first order ODE in standard form with $p(x) = q(x) = x$.

Solution:

$$\mu(x) = e^{\int x\,dx} = e^{\frac{x^2}{2}}$$

$$\Rightarrow \quad \frac{d}{dx}\left(e^{x^2/2} y\right) = x e^{x^2/2}$$

and after integration

$$e^{x^2/2} y = \int x e^{x^2/2}\,dx + C = e^{x^2/2} + C$$

$$\Rightarrow \quad y = 1 + C e^{-x^2/2}$$

For the initial condition $y(0) = 1$ we get

$$y = 1, \quad (C = 0);$$

and for $y(0) = a$ we get

$$y = 1 + (a-1)e^{-x^2/2}, \quad (C = a-1).$$